Wigner distributions for finite dimensional quantum systems: An algebraic approach

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Abstract

We discuss questions pertaining to the definition of 'momentum', 'momentum space', 'phase space', and 'Wigner distributions'; for finite dimensional quantum systems. For such systems, where traditional concepts of 'momenta' established for continuum situations offer little help, we propose a physically reasonable and mathematically tangible definition and use it for the purpose of setting up Wigner distributions in a purely algebraic manner. It is found that the point of view adopted here is limited to odd dimensional systems only. The mathematical reasons which force this situation are examined in detail.

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1. INTRODUCTION

The description of states of quantum systems with the help of Wigner distributions, in the case of continuous Cartesian position and momentum variables [1]-[3], has found applications in a wide variety of contexts. These include the original aim of calculating quantum corrections to classical statistical mechanics, various aspects of quantum optics, and quantum chemistry [4].

There has been for some time considerable interest in developing a similar method for quantum systems possessing a finite dimensional state space [5]-[12]. In the works of Feynman[6] and of Wootters[5], which constitute some of the early efforts in this direction, it was found necessary to treat independently the cases of two-dimentional state space, and of an odd prime dimensional state space. In the former, corresponding to the case of a qubit, the insights from the treatment of a spin 1/2 angular momentum in quantum mechanics were used effectively. The treatment of the odd prime dimension case has been geometric in spirit. From the start it is assumed that the phase space is a square array of points, namely that 'momentum space' is of the same 'size' as 'position' or configuration space. In this phase space, the notions of straight lines, parallel lines, and foliations or striations are set up and used in stating the properties desired of a Wigner distribution and then a solution is proposed [5],[11]. The arithmetic of a finite field of odd prime order is used in the constructions. For general odd dimensions or a power of two, one has to take the Cartesian product of the elementary solutions.

The purpose of the present work is to provide an alternative treatment of this problem based on an algebraic approach. As we will find, apart from reproducing known results (in the odd dimension cases), essentially new possibilities will also come to light. Our work is motivated by the following points of view. In elementary mechanics, the primitive notion of momentum is as a numerical measure of rate of change of position in space, the factor of proportionality being the mass. In canonical mechanics we move to an algebraic role for momentum, as generator of translations in configuration space, with mass not playing an explicit role. In Cartesian quantum mechanics this algebraic role is expressed by the canonical Heisenberg commutation relations and is fundamental.

Returning to classical canonical mechanics usually the configuration space Q is taken to be a differentiable manifold of some dimension. Then the concepts of momenta and phase

space are given immediately by the automatic construction of the cotangent bundle $T^\star Q$. There are of course special features in the Cartesian case $Q = \mathbb{R}^n$, not shared by the general case. Then $T^\star Q = \mathbb{R}^{2n}$, and both positions and conjugate momenta are elements of real linear vector spaces of the common dimension, n. Thus they are of the same algebraic nature, allowing us to form real linear combinations of them in a physically significant manner. This leads to an important role for the groups $Sp(2n, R)$ in the classical case, and to the double covers $Mp(2n)$ after quantisation. For a general configuration space Q different from $\mathbb{R}^n$, for instance $\mathbb{S}^n$ or the manifold of a Lie group[13], one sees that coordinates and momenta have intrinsically different algebraic natures. Locally momenta always belong to a linear vector space, while positions may have no such characterisation. In these cases, of course, quantum mechanics uses suitably square integrable functions on Q as wave functions, and the basic operator relations are inspired by what one knows about the classical $T^\star Q$. However, already in the simple case of $Q = \mathbb{S}^2$, a particle confined to the surface of the unit sphere, one sees an interesting new possibility emerging. Since $T^\star \mathbb{S}^2$ is not parallelisable, the canonical momenta cannot be globally defined as variables. However one can view $\mathbb{S}^2$ as the coset space $SO(3)/SO(2)$, and so think of canonical momenta as the constrained components of a three dimensional angular momentum, obeying the algebraic requirement of being orthogonal to the (unit) position vector. This makes the momenta globally well defined, besides bringing into the picture a role for the group $SO(3)$ acting transitively on $\mathbb{S}^2$.

We now turn to an N-level quantum system. The state space is a complex N-dimensional Hilbert space, wave functions are complex N-component column vectors, but properly speaking no notion of canonical momenta is given in advance in any intrinsic sense. There are no fundamental operator relations analogous to the canonical Heisenberg commutation relations. In this case, classically we may view the configuration space Q as a finite set of N discrete points, without any concept of canonical momenta. This leads to the viewpoint: for finite level systems, the problems of defining momenta, momentum space and phase space should be regarded as part of the problem of setting up Wigner distributions in a physically reasonable manner.

From this discussion we extract and explore the idea that for any choice of configuration space Q, whether a differentiable manifold or a finite discrete set, the concept of momentum is <u>any group</u> G <u>that acts transitively on</u> Q. In case Q is a differentiable manifold and G acts on Q via diffeomorphisms, we move a step closer to the usual meaning of momentum by

consideration of the generators of these diffeomorphisms subject to as many linear algebraic constraints as the difference in the dimensions of G and Q.

This general viewpoint means that Q is a coset space G/H with respect to some subgroup $H \subset G$. We will however assume for definiteness that H is trivial, i.e., $Q = G$. Thus the configuration space is itself a group. We will find that using the representation theory of G, in the quantum case this leads to an algebraic way of defining momenta and phase space. However when we turn to the problem of defining Wigner distributions the method is limited to odd dimensions only for a reason that will emerge later.

This work is organised as follows. In Section 2, following the line of thought outlined above, we take a group G of order N as the 'classical' configuration space underlying an N-level quantum system; and proceed to set up the necessary machinery for introducing the notions of 'positon space', 'momentum space' and 'a point in phase space', recapitulating, in the process, relevant aspects of the representation theory of finite groups. A noteworthy feature of the phase space thus obtained is that it is always an $N \times N$ array for any N regardless of the nature of G. Having developed the notion of 'phase space' for an N-level quantum system, for any N, in Section 3, we take up the question of constructing a Wigner distribution on such a phase space. This requires introducing the concept of a 'square root' of a group element, which, as it turns out, is possible only when N is odd. Restricting ourselves to such dimensions, we show that, in general, there are two equally good ways of defining Wigner distributions which coincide only when G is an abelian group. Choosing one of them, and keeping G general, we construct the corresponding phase point operators [5],[14] and find, in particular, that the phase point operator at the origin, like its continuum counterpart, has the satisfying feature of being an inversion operator. We then discuss in some detail the special case when G is abelian and analyse the two possibilities (a) N an odd prime (b) N an odd composite. While in the former case, G gets uniquely fixed to be a cyclic group of order N, in the latter case, in general, one has more than one choice for G leading to distinct definitions of the Wigner distribution. (The material presented in this section makes use of the results in [12],[13] though with some changes of notation). In Section 4 we turn to the case when G is nonabelian and analyse the relation between the two choices for the Wigner distribution. We further show that it is possible to define an 'extended' Wigner distribution which contains the two Wigner distributions as its marginals. However, as is explicitly demonstrated, this can only be achieved at the cost of introducing

overcompleteness. Section 5 contains our concluding remarks and further outlook.

2. FINITE ORDER CONFIGURATION AND PHASE SPACES

Our aim is to describe general states of an N-level quantum systems (for odd N) via Wigner distributions. The space of (pure) quantum states is an N-dimensional complex Hilbert space $\mathcal{H}^{(N)}$, each vector $|\psi>\in\mathcal{H}^{(N)}$ being an N-component complex column vector. The corresponding or comparison classical configuration space Q is a finite discrete set of N points. We wish to regard Q as a finite group G of order N. This makes it natural to label the points of Q as $g,\ g',\cdots,\ a,\ b,\cdots\in G$, with a distinguished 'origin' corresponding to the identity element $e\in G$. In turn in the quantum case we introduce an orthonormal basis of 'position eigenstates' $|g>$ for $\mathcal{H}^{(N)}$:

$$\begin{aligned}&< g'|g>=\delta_{g',g},\quad g,\ g'\in G;\\&\sum_{g\in G}|g><g|=\mathbb{I}.\end{aligned}\tag{2.1}$$

A general $|\psi>\in\mathcal{H}^{(N)}$ has a position space wave function

$$\begin{aligned}&\psi(g)=< g|\psi>,\\&<\psi|\psi>=||\psi||^2=\sum_g|\psi(g)|^2.\end{aligned}\tag{2.2}$$

To move onto the concepts of momentum and phase space, we consider the transitive action of G on $Q=G$, namely of G on itself. Keeping in mind the general non-abelian situation, we have two such actions, namely left translations L_g and right translations R_g:

$$\begin{aligned}(L_{g'}\psi)(g)&=\psi(g'^{-1}g),\\L_{g'}|g>&=|g'\ g>;\\(R_{g'}\psi)(g)&=\psi(g\ g'),\\R_{g'}|g>&=|g\ g'^{-1}>.\end{aligned}\tag{2.3}$$

(In the abelian case, $L_g=R_g$). Both L_g and R_g are unitary, and they obey

$$\begin{aligned}L_{g'}\ L_g&=L_{g'g},\\R_{g'}\ R_g&=R_{g'g},\\L_{g'}\ R_g&=R_g\ L_{g'}.\end{aligned}\tag{2.4}$$

Thus we have two commuting representations-the regular representations of G on $\mathcal{H}^{(N)}$. We arrive at the notion of 'momentum eigen states' by performing their simultaneous complete reduction into irreducibles. For this we recall well known facts regarding the inequivalent unitary irreducible representations (UIR's) of G. We label them with a symbol j, the corresponding dimension being N_j. With respect to some chosen orthonormal basis within each UIR, we have corresponding unitary representation matrices

$$D^j(g) = \left(D^j_{mn}(g)\right), \quad m, n = 1, 2, \cdots, N_j. \tag{2.5}$$

The following well known properties and facts will be needed:

$$\begin{aligned}
&\text{Composition} & \sum_{n'} D^j_{mn'}(g')\; D^j_{n'n}(g) \;&=\; D^j_{mn}(g'g); && \text{(2.6a)}\\
&\text{Unitarity} & \sum_{n} D^j_{mn}(g)\; D^j_{m'n}(g)^* \;&=\; \delta_{mm'}; && \text{(2.6b)}\\
&\text{Orthogonality} & \sum_{g} D^j_{mn}(g)\; D^{j'}_{m'n'}(g)^* \;&=\; \frac{N}{N_j}\; \delta_{jj'}\; \delta_{mm'}\; \delta_{nn'}; && \text{(2.6c)}\\
&\text{Completeness} & \sum_{jmn} N_j\; D^j_{mn}(g)\; D^j_{mn}(g')^* \;&\equiv\; \sum_{jm} N_j\; D^j_{mm}(gg'^{-1}) && \\
& & &=\; N\delta_{g,g'}; && \text{(2.6d)}\\
& & \sum_j N_j^2 \;&=\; N. && \text{(2.6e)}
\end{aligned}$$

It is now natural to define an orthonormal basis of 'momentum eigenstates' for $\mathcal{H}^{(N)}$, complementary to the basis $\{|g>\}$, as vectors labelled $|jmn>$:

$$\begin{aligned}
|jmn> \;&=\; \sqrt{\frac{N_j}{N}} \sum_g D^j_{mn}(g)|g>,\\
< g|jmn> \;&=\; \sqrt{\frac{N_j}{N}}\; D^j_{mn}(g).
\end{aligned} \tag{2.7}$$

They obey

$$\begin{aligned}
< j'm'n'|jmn> \;&=\; \delta_{j'j}\; \delta_{m'm}\; \delta_{n'n},\\
\sum_{jmn} \; |jmn < jmn| \;&=\; \mathbb{I},
\end{aligned} \tag{2.8}$$

while the completely reduced actions of L_g and R_g are :

$$\begin{aligned}
L_g\; |jmn> \;&=\; \sum_{m'} \; D^j_{mm'}(g^{-1})\; |jm'n>,\\
R_g\; |jmn> \;&=\; \sum_{n'} \; D^j_{n'n}(g)\; |jmn'> .
\end{aligned} \tag{2.9}$$

Therefore 'momentum' is 'conjugate' to 'position' in the sense that when jmn and g are given, a complex number $\sqrt{N_j/N}\ D^j_{mn}(g)$ is determined.

A vector $|\psi> \in \mathcal{H}^{(N)}$ thus has two complementary wave functions: $\psi(g)$ of eq.(2.2), and a momentum space wave function

$$\psi_{jmn} = < jmn|\psi > . \tag{2.10}$$

If $|\psi>$ is normalised, then it determines two complementary probability distributions, as

$$||\psi||^2 = \sum_g |\psi(g)|^2 = \sum_{jmn} |\psi_{jmn}|^2 = 1. \tag{2.11}$$

For purposes of setting up Wigner distributions, we thus regard a 'point in phase space' as a pair $(g; jmn)$. By eq.(2.6e) we then find that the 'size' of momentum space is precisely N, the same as the 'size' of Q: so the points $(g; jmn)$ of phase space form a square array. We obtain this result here not as a matter of definition but as a consequence of an important result of group representation theory. In the phase space there is a distinguished 'origin' corresponding to $(e; 000)$: here $g = e$, and $j = m = n = 0$ denote the trivial or identity representation of G.

The developments of this section so far are valid for all N. Further, each way of regarding Q as a group of order N leads to a corresponding way of building up momenta and phase space. Thus if G and G' are two non isomorphic groups of order N, they lead to two distinct notions of momenta and phase space.

3. WIGNER DISTRIBUTIONS

We begin by recalling an important property available only for odd N, i.e., a property of finite groups of odd order. It is that in such a group each element $g \in G$ has a unique 'square root' element written as $\sqrt{g}$ or $g^{1/2}$:

$$g \in G \Longrightarrow \sqrt{g} \in G \quad : \quad (\sqrt{g})^2 = g. \tag{3.1}$$

The uniqueness and other useful properties of this notion are as follows:

$$g_1 = g_2 \iff \sqrt{g_1} = \sqrt{g_2}, \quad g_1^2 = g_2^2; \tag{3.2a}$$

$$(\sqrt{g})^{-1} = \sqrt{g^{-1}}; \tag{3.2b}$$

$$g\sqrt{g} = \sqrt{g}g; \tag{3.2c}$$

$$g^{-1}\sqrt{g} = \sqrt{g^{-1}}; \tag{3.2d}$$

$$\sqrt{gg'g^{-1}} = g\sqrt{g'}g^{-1}. \tag{3.2e}$$

Therefore in any sum over G of a function $f(g)$ belonging to a linear space we have the freedom to write the sum in different ways:

$$\sum_g f(g) = \sum_g f(\sqrt{g}) = \sum_g f(g^2). \tag{3.3}$$

A further consequence is :

$$a = g'^{-1}g, \quad b = g'g \iff$$
$$g' = \sqrt{ba^{-1}}, \quad g = \sqrt{ab^{-1}}b = \sqrt{ba^{-1}}\ a; \tag{3.4a}$$
$$a = gg'^{-1}, \quad b = gg' \iff$$
$$g' = \sqrt{a^{-1}b}, \quad g = b\sqrt{b^{-1}a} = a\sqrt{a^{-1}b}. \tag{3.4b}$$

In both cases, for given a and b, g and g' are unique. In the nonabelian case, these two equations differ, while in the abelian case they coincide. We will find that these two properties of any group of odd order are essential to be able to define Wigner distributions with desirable properties using the present method.

We have seen in Section 2 that a point in phase space is a pair $(g; jmn)$. It is then natural to look for a Wigner distribution in the form $W_\psi(g; jmn)$ for given normalised $|\psi> \in \mathcal{H}^{(N)}$ and require that it have desirable properties with regard to reality, transformations under left and right group actions, reproduction of the two marginal probability distributions in eq.(2.11), and traciality. Keeping in mind the general case of nonabelian G, it turns out that there are two distinct but equally good ways of setting up Wigner distributions, which reproduce complementary features of the momentum space probability distribution $|\psi_{jmn}|^2$. We define them in sequence.

Wigner distribution I

For any given normalised $|\psi>\in\mathcal{H}^{(N)}$ we define a corresponding Wigner distribution by

$$W_\psi(g;jmm') = \sum_{g'} \psi(g'^{-1}g) D^j_{mm'}(g'^{\,2}) \psi(g'g)^*. \tag{3.5}$$

This definition extends immediately to a general state $\widehat{\rho}$ as

$$W_{\widehat{\rho}}(g;jmm') = \sum_{g'} < g'^{-1}g|\widehat{\rho}|g'g > D^j_{mm'}(g'^{\,2}), \tag{3.6}$$

however for simplicity we shall work mainly with the pure state case. Using the properties $(2.6, 3.3, 3.4)$ as necessary, we find :

$$\begin{aligned}
W_\psi(g;jmm')^* &= W_\psi(g;jm'm); && (3.7a)\\
\psi' = L_{g'}\psi : W_{\psi'}(g;jmm') &= \sum_{m_1,m_1'} D^j_{mm_1}(g') W_\psi(g'^{-1}g;jm_1m_1') \times \\
&\qquad\qquad D^j_{m_1'm'}(g'^{-1}), \\
\psi'' = R_{g'}\psi : W_{\psi''}(g;jmm') &= W_\psi(gg';jmm'); && (3.7b)\\
\frac{1}{N}\sum_g N_j\ W_\psi(g;jmm') &= \sum_n \psi^*_{jmn}\psi_{jm'n}\ , \\
\frac{1}{N}\sum_{jm} N_j\ W_\psi(g;jmm) &= |\psi(g)|^2; && (3.7c)\\
\frac{1}{N}\sum_g\sum_{jmm'} N_j\ W_\phi(g;jm'm) \qquad & W_\psi(g;jmm') = |<\phi|\psi>|^2. && (3.7d)
\end{aligned}$$

From the last traciality property it is clear that $|\psi>$ is completely described by $W_\psi(g;jmm')$ as far as all quantum mechanical probabilities are concerned. For the marginals, we see that the 'position space' probability distribution $|\psi(g)|^2$ is faithfully reproduced. However, in the nonabelian case, the 'momentum space' probability distribution (in a polarised form) is reproduced in what we may call a 'coarse grained' version. This is further discussed in Section 4.

Properties $(3.7a, 3.7b)$ and the second of (3.7c) are evident essentially upon inspection. The first of eq.(3.7c) follows upon use of eq.(3.4a) and changing summation variables. For the traciality property (3.7d) slightly more work is needed. Using eq.(2.6d) at the appropriate

stage, and then eq.(3.4a), we get:

$$
\begin{aligned}
&\frac{1}{N}\sum_{g}\sum_{jmm'} N_j W_\phi(g;jm'm)W_\psi(g;jmm') = \\
&\frac{1}{N}\sum_{g,g',g''}\sum_{jmm'} N_j\phi(g'^{-1}g)D^j_{m'm}(g'^{\,2})\phi(g'g)^*\psi(g''^{-1}g)D^j_{mm'}(g''^{\,2})\psi(g''g)^* \\
= &\sum_{g,g',g''}\phi(g'^{-1}g)\psi(g''g)^*\psi(g''^{-1}g)\phi(g'g)^*\delta_{g'',g'^{-1}} \qquad (3.8)
\end{aligned}
$$

which immediately gives (3.7d).

Wigner distribution II

Now in place of eq.(3.5) we define a different (in the nonabelian case !) Wigner distribution by

$$
W_\psi{}'(g;jnn') = \sum_{g'}\psi(gg'^{-1})D^j_{nn'}(g'^{\,2})\psi(gg')^*. \qquad (3.9)
$$

Then in place of eqs.(3.7) we find :

$$
\begin{aligned}
W_\psi{}'(g;jnn')^* &= W_\psi{}'(g;jn'n); \qquad (3.10a)\\
\psi' = L_{g'}\psi : W'_{\psi'}(g;jnn') &= W_\psi{}'(g'^{-1}g;jnn'),\\
\psi'' = R_{g'}\psi : W'_{\psi''}(g;jnn') &= \sum_{n_1,n_1'} D^j_{nn_1}(g')W_\psi{}'(gg';jn_1n_1')D^j_{n_1'n'}(g'^{-1});\\
&\qquad (3.10b)\\
\frac{1}{N}\sum_g N_jW_\psi{}'(g;jnn') &= \sum_m \psi_{jmn}\psi^*_{jmn'},\\
\frac{1}{N}\sum_{jn} N_jW_\psi{}'(g;jnn) &= |\psi(g)|^2; \qquad (3.10c)\\
\frac{1}{N}\sum_g\sum_{jnn'} N_jW_\phi{}'(g;jn'n) \qquad & W_\psi{}'(g;jnn') = |<\phi|\psi>|^2. \qquad (3.10d)
\end{aligned}
$$

Having exhibited the existence of these two ways of defining the Wigner distribution distinct for nonabelian G, we hereafter mainly work with definition I, except in Section 4 where we explore the relationships between the two.

Phase Point Operators

In analogy with earlier treatments, it is possible to express the Wigner distribution (3.5)

as the expectation value of a 'phase point operator' in the state $|\psi>$:

$$\begin{aligned} W_\psi(g;jmm') &= <\psi|\widehat{W}(g;jmm')|\psi>, \\ \widehat{W}(g;jmm') &= \sum_{g'} |g'g> D^j_{mm'}(g'^{\,2}) < g'^{-1}g|. \end{aligned} \tag{3.11}$$

Since with the help of eq.(2.6d) and (3.2a) we are able to 'invert' this to read

$$|g'g>< g'^{-1}g| = \sum_{jmm'} \frac{N_j}{N} D^j_{mm'}(g'^{\,2})^* \widehat{W}(g;jmm'), \tag{3.12}$$

we see that they provide a basis for the space of all operators on $\mathcal{H}^{(N)}$. They have the specific properties:

$$\widehat{W}(g;jmm')^\dagger = \widehat{W}(g;jm'm); \tag{3.13a}$$

$$\frac{N_j}{N}\sum_g \widehat{W}(g;jmm') = \sum_n |jmn>< jm'n|; \tag{3.13b}$$

$$\sum_{jm} \frac{N_j}{N}\widehat{W}(g;jmm) = |g>< g|; \tag{3.13c}$$

$$\mathrm{Tr}[\widehat{W}(g;jmm')] = \delta_{mm'}; \tag{3.13d}$$

$$\mathrm{Tr}[\widehat{W}(g;jmn)\ \widehat{W}(g';j'm'n')^\dagger] = \frac{N}{N_j}\delta_{g,g'}\delta_{jj'}\delta_{mm'}\delta_{nn'}; \tag{3.13e}$$

which are easily obtained by eqs.(3.4a, 2.7, 2.6d) as appropriate.

We noted earlier that the point $(e;000)$ in phase space may be regarded as a distinguished 'origin'. The corresponding phase point operator $\widehat{W}(e;000)$, from eq.(3.11), has the simple form and action

$$\begin{aligned} \widehat{W}(e;000) &= \sum_{g'} |g'>< g'^{-1}|, \\ \widehat{W}(e;000)|g> &= |g^{-1}> . \end{aligned} \tag{3.14}$$

We may call this as the inversion operation. This is the analogue, in the present finite dimensional case treated algebraically, of the known result in the case of continuous variables that the phase point operator at the origin of phase space is a multiple of the parity operator.

The Odd Prime Case

In case N is an odd prime, the only choice for G is to be an abelian cyclic group of order N, so we exhibit it as

$$\begin{aligned} G &= \{e, g, g^2, \cdots g^{N-1}|g^N = e\} \\ &= \{g^n|n = 0, 1, \cdots, N-1\}. \end{aligned} \tag{3.15}$$

The UIR's are all one dimensional labelled by an integer $j = 0, 1, 2, \cdots, N-1$:

$$\begin{aligned} D^j(g) &= e^{2\pi i j/N}, \\ D^j(g^n) &= e^{2\pi i j n/N}. \end{aligned} \tag{3.16}$$

The unique square root of g^n is easily computed:

$$\begin{aligned} \sqrt{g^n} &= g^{(n+N)/2}, \quad n = 1, 3, 5, \cdots, N-2; \\ &= g^{n/2}, \qquad\quad n = 2, 4, 6, \cdots, N-1\ . \end{aligned} \tag{3.17}$$

The two Wigner distribution definitions (3.5, 3.9) coincide:

$$W_\psi(g^n; j) = \sum_{n'=0}^{N-1} \psi(g^{n-n'}) e^{2\pi i j.2n'/N} \psi(g^{n+n'})^*, \tag{3.18}$$

reproducing a known result [5]. There is no 'coarse graining' involved in the recovery (3.7c) of the marginal momentum space probability distribution.

For general odd N, we may have again a unique G in the abelian case, or more than one distinct abelian choice. For example, for $N = 9$, the two choices $G = C_9$ and $G' = C_3 \times C_3$, where C_n is the cyclic group of order n, are nonisomorphic, so they lead to distinct definitions of the Wigner distribution. On the other hand for $N = 15$, C_{15} and $C_3 \times C_5$ are isomorphic. The general pattern can be seen along these lines.

The first case of a nonabelian G occurs when $N = 21$, some others occur for instance when $N = 27, 39, 45, 55, 57, 63, 75, \cdots$. It is in these cases that the 'coarse graining' of the momentum space probability distribution seen in eqs.(3.7c, 3.10c) in two different ways, is involved. We examine this in some detail in the next section.

4. ANALYSIS OF THE NONABELIAN CASE

Let N be an odd nonprime such that a nonabelian group G of order N exists. In that case some further interesting properties of the two definitions of Wigner distributions given in Section 3 can be exhibited. To begin with, by a simple change of variable $g' \to g^{-1} g' g$ in equation eq.(3.9) we can put the second definition into the form

$$W_\psi{}'(g; jnn') = \sum_{g'} \psi(g'^{-1} g) D^j_{nn'}(g^{-1} g'^{\,2} g) \psi(g' g)^*. \tag{4.1}$$

Comparing with $W_\psi(g;jmm')$ in eq.(3.5), and for each g and j treating the two Wigner distributions as $N_j \times N_j$ matrices, we see that

$$W_\psi{}'(g;j) = D^j(g^{-1})W_\psi(g;j)D^j(g), \tag{4.2}$$

making clear the relation between them in detail.

Next we turn to the coarse graining of the momentum space probability distribution involved in eqs.(3.7c, 3.10c). It is in fact possible to define an overcomplete or redundant Wigner distribution in the following manner:

$$\begin{aligned} w_\psi(g;jmn'\ m'n) &= \sum_{g'} \psi(g'^{-1}g)D^j_{mn'}(g'g)D^j_{nm'}(g^{-1}g')\psi(g'g)^* \\ &= \sum_{g'} \psi(gg'^{-1})D^j_{mn'}(gg')D^j_{nm'}(g'g^{-1})\psi(gg')^*. \end{aligned} \tag{4.3}$$

From here the two actual Wigner distributions emerge by partial tracing:

$$\begin{aligned} W_\psi(g;jmm') &= \sum_n w_\psi(g;jmn\ m'n), \\ W_\psi{}'(g;jnn') &= \sum_m w_\psi(g;jmn'\ mn). \end{aligned} \tag{4.4}$$

The quantities $w_\psi(g;jmn'\ m'n)$ have the properties

$$\begin{aligned} w_\psi(g;jmn'\ m'n)^* &= w_\psi(g;jm'n\ mn'); \\ \frac{N_j}{N}\sum_g w_\psi(g;jmn'\ m'n) &= \psi_{jm'n}\psi^*_{jmn'}; \\ \frac{1}{N}\sum_{jmn} N_j w_\psi(g;jmn\ mn) &= |\psi(g)|^2. \end{aligned} \tag{4.5}$$

These results are consistent with eqs.(4.4, 3.7, 3.10), and the behaviours under left and right group actions can be easily worked out. We now see that all the momentum space probabilities (in a polarised form), without any coarse graining, can be recovered from the distribution $w_\psi(g;jmn'\ m'n)$. However this distribution is overcomplete, as is demonstrated by showing explicitly that it can be reconstructed from, say, $W_\psi(g;jmm')$. Combining the definition (4.3) with eqs.(3.11, 3.12) we have:

$$\begin{aligned} w_\psi(g;jmn'\ m'n) &= \sum_{g'} <\psi|g'g><g'^{-1}g|\psi> D^j_{mn'}(g'g)D^j_{nm'}(g^{-1}g') \\ &= \sum_{g'}\sum_{j''m''n''} \frac{N_{j''}}{N} D^{j''}_{m''n''}(g'^{\,2})^* D^j_{mn'}(g'g)D^j_{nm'}(g^{-1}g') \times \\ &\qquad\qquad W_\psi(g;j''m''n''). \end{aligned} \tag{4.6}$$

Again it is easily checked that this is consistent with eqs. $(4.2, 4.4)$. This clarifies the situation in the nonabelian case. The full momentum space probability distribution $|\psi_{jmn}|^2$, more generally the polarised expressions $\psi_{jm'n}\psi^*_{jmn'}$, are all obtainable in principle from the Wigner distribution $W_\psi(g; jmm')$, even though at first sight eq.(3.7c) seems to suggest that only the coarse grained expressions $\sum_n \psi_{jm'n}\psi^*_{jmn}$ can be recovered. It is indeed the case that $W_\psi(g; jmm')$ describes $|\psi><\psi|$ completely.

5. CONCLUSIONS

In this work we have proposed a way of associating a phase space with an N-level quantum system by taking the corresponding 'classical' or comparison configuration space as consisting of the N elements of a group G of order N. A gratifying feature of our construction that emerges is that the phase space always consists of an $N \times N$ array independently of the nature of the group G. We have explored the possibilities for defining Wigner distributions on such phase spaces and have shown that this turns out to be possible only when N is odd. Further, we have shown that for such dimensions, there exist two equally good choices for the Wigner distribution and have discussed the relation between them in detail. We have also shown that the two Wigner descriptions can be put together into an extended Wigner distribution from which they can be recovered by partial tracing. Such a synthesis, however, can be achieved only at the expense of introducing overcompleteness into the description.

The most curious feature that emerges from this work is the importance of being odd. We hope to return to this and related questions elsewhere.

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